A stochastic, geostatistic and reliability view on some geotechnical distributions

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Abstract

In this paper the following problems are treated:
- Estimation of the mean value of a random function $Z(x)$, defined in a stochastic finite element $v$, (SFE),

$$z_v = \frac{1}{v} \int_v Z(x)dx,$$

where the distributions of $Z(x)$ at each node are known;
- Kriking solution with SFE under the non-stationary hypothesis:

$$E(Z(x)) = m(x), \quad C(x, h) = E\{(Z(x+h)Z(x))-m(x+h)m(x).$$

Finally are given the conclusions underlying the importance of above stochastic instruments not only in the stochastic geotechnical discipline but also in other ones as in energy, geology, geophysics, mechanics, dynamics, elastostatics, finance, engineering, environment, climate etc., in which the distributions are used.

1. Estimation of the mean value of a random function $Z(x)$, defined in a stochastic finite element (SFE) $v$,

$$z_v = 1/v \int_v Z(x)dx,$$

where the distributions of $Z(x)$ at each node are known;

2. A discretization random field view of SFE in relation to other discretized methods.

3. Kriking in SFE view

4. SFE in reliability analysis.

5. Finally some considerations are presented, related to stochastic random field proprieties estimation and stochastic differential equations.

Introduction

The random theory [1] seeks to model complex patterns of variation and interdependences in cases where deterministic treatments are inefficient and conventional statistics are insufficient
In this view stochastic calculus [2],[4],[13],[14],[15],[16]. on input, output, system models etc are the key points in the random field theory and its applications in geotechnics.

It is known, a random field \( X(t) \) is defined as a collection of random variables indexed by a continuous parameter \( t \in \mathbb{R}^d \), \( (d = 2 \text{ or } 3) \), i.e. \( t \) is a location vector. To completely define a random field, the joint distribution of the random variables \( \{X(t_1), X(t_2), \ldots, X(t_n)\} \) for any \( \{n, t_1, t_2, \ldots, t_n\} \) must be specified.

The continuous random field \( X(t) \) as soil, elastic etc medium may be approximated by a discrete, defined in terms of a finite set of random variables \( \{X_1, X_2, \ldots, X_n\} \). Like in the deterministic FE method, the stochastic discretization method requires the splitting of the domain into a discrete assembly of elements, which is usually referred to as the stochastic finite element (SFE) mesh. All the methods found in the literature under this denomination have the following common characteristics:

– a finite element model, i.e. the discretized version of the equations governing a physical phenomenon etc.

– a probabilistic model of the input parameters: random variables and/or random fields. In this context below it is presented a SFE method.

**Stochastic finite element. the mean value**

Let’s consider: a zone \( V \subset \mathbb{R}^3 \) and a random function \( Z(x) \), \( x \in V \); the zone \( V \) is partitioned into blocks \( v_i \) by a parallelepiped grid:

\[
V = \bigcup v_i
\]

where \( v_i \) is a parallelepiped element with eight nodes. In each node \( Z(x) \) is known i.e. the probability density on its point support as it is shown in Fig(1). It is required:

• the distribution p.d.f [16],[9] in whatever point \( x \in V \)

• the estimation of the mean value [16],[9]]

\[
z_{vi} = 1/v_i \int_v Z(x)dx \tag{2}
\]

Fig. 1. Parallelepiped element

over the domain \( v_i \).
We define a stochastic element as a block [12], with the random function \( Z(x) \), \( x \in \mathcal{V}_i \), so \( Z(x) \) at a point \( x \in \mathcal{V}_i \) is a random variable with the respective (probability) distribution. In fig 1a are presented four distributions well logs.

Let us consider a reference element \( w_i \) in the coordinate system \( s_1 \ s_2 \ s_3 \). If we choose an incomplete base [16]:

\[
P(s) = \langle 1 \ s_1 \ s_2 \ s_3 \ s_1s_2 \ s_2s_3 \ s_3s_1 \ s_1s_2s_3 \rangle
\]

then the function \( Z(x) \) could be presented as a linear combination:

\[
Z(x) = Z( s_1 \ s_2 \ s_3 ) = < P(s) > [ P_8 ]^{-1} \{ Z_s \} = < N(s) > \{ Z_s \}
\]

\([P_8]^{-1} \) - the matrix, whose elements are the polynomials base values at the nodes, \( \{ Z_s \} \) is the vector of the distributions of the nodes, while \( < N(s) > \) is the vector of the shape functions \( N_i, i = 1,8 \)

\[
N(s) = < N_i(s) > = < N_1(s), N_2(s), \ldots, N_8(s) >
\]

where

\[
N_i(s) = \frac{1}{8} (1+s_1s_i^i) (1+s_2s_i^i) (1+s_3s_i^i) \quad i=1,8
\]

To calculate the mean value \( z_{vi} = 1/\mathcal{V}_i \int_{\mathcal{V}_i} Z(x)dx \), we consider the deterministic transformation

\[
X_i(s) = < N_i(s) > < x_i^s > \quad i=1,3
\]

Therefore
\[ Z_{v} = \frac{1}{V} \int Z(x_1(s_1, s_2, s_3), x_2(s_1, s_2, s_3), x_3(s_1, s_2, s_3)) \det J \, ds_1 \, ds_2 \, ds_3 = \sum_{i=1}^{8} H_i Z(x) \]

After some algebra [16], we can calculate the weight coefficient \( H_i \), \( i=1,8 \), for example

\[ H_2 = \frac{8}{3} c_{32} a_{13} + \frac{8}{9} d_{21} a_{12} a_{13} + \frac{8}{27} c_{32} d_{13} + \frac{8}{27} d_{21} c_{32} b_{13} + \frac{8}{9} a_{21} b_{32} c_{32} + \frac{8}{9} b_{21} b_{32} a_{13} + \frac{8}{9} c_{32} b_{32} c_{32} \]

The analogous calculations it could be done [18] for the other coefficient \( H_i \), \( i=1,8 \). Also it can easily be proved

\[ \sum_{i=1}^{8} H_i = 1 \]

then

\[ E\{ Z_v(x)\} = \sum_{i=1}^{8} H_i E\{ Z_i(x)\} = \sum_{i=1}^{8} H_i m = m, \quad \forall i=1,8 \quad (9) \]

where the coefficients \( H_i \) are the distribution weights.

It is shown [16],[13], the stochastic finite element estimator is a linear interpolator, related to the distributions given at its nodes and it could be calculated using Monte Carlo simulations. Generally SFE, there are many applications in geotechnics [5] as in random fields discretization, stochastic integrals, differential equations(SDE), geostatistics, reliability theory, rock elasticity-plasticity linear theory etc. Of course the mention approach differs to other ones, for example to the midpoint method, the shape function method, the integration point method which use other bases to approximate a given random field.

**Kriging in sfe view**

Kriging refers to a group of generalized least-squares regression algorithms. Kriging refers to a group of generalized least-squares regression algorithms). All kriging methods are variants of the linear regression estimator [3], given by:

\[ Z^*(u) = m(u) + \sum_{\alpha=1}^{n(u)} \hat{\lambda}_\alpha(u)[Z(u_{\alpha}) - m(u_{\alpha})] \]

(10)

where \( \hat{\lambda}_\alpha(u) \) is the weight assigned to datum \( Z(u_{\alpha}) \). \( m(u) \) and \( m(u_{\alpha}) \) are the expected values of the random variables defined for each location.

Taking account the the mean value
\[ Z_v = \frac{1}{v} \int_Z Z(x) \, dx \quad \text{(11)} \]

over a given domain \( v \) and the Kriking definition [17].

Above SFE could be used for calculus of Kriking variants [3], [11]. Simple Kriking, Ordinary kriking, Trend only kriking, Universal kriking, Lognormal kriking, Indicator kriking, Cokriking etc.

Kriking algorithm with SFE could also be worth for solving stochastic partial differential equations under some conditions - to approximate “their stochastic coefficients” by mean values in conformity with SFE presented.

- to consider the differential equations in mean square sense. Remember, a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is called mean square differentiable in \( x_0 \) in direction \( x \) when the differential quotient

\[
\lim_{h \to 0} \frac{f(x_0 + hx) - f(x_0)}{h}
\]

has a limit of the mean and the variance for \( h \to 0 \). In this view the theorem could be considered:

If:

- a random function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is a mean square differentiable in \( x_0 \),

\[
\nabla_x f(x) := \left( \frac{\partial}{\partial x_1} f(x), \ldots, \frac{\partial}{\partial x_d} f(x) \right)'
\]

exists and has finite variation

- the trend function \( g_1(x), g_2(x), \ldots, g_p(x) \) solve the homogeneous equation

\[
L_x g_i(x) = 0 \forall i = 1, \ldots, p
\]

(12)

2- THE TREND PART \( G_0(x) \) SOLVES THE HETEROGENEOUS DIFFERENTIAL EQUATION

\[
L_x G_0(x) = k(x).
\]

(13)

- THE COVARIATION EXISTS AND IS ADMISSIBLE, I.E.

\[
L_x L_y c(x, y)|_{x=y} = 0,
\]

(13)

for the linear partial differential equation

\[
L_x f(x) = k(x)
\]

then

its solution is the universal kriking interpolation
SFE is a appropriate tool not only to calculate the covariation matrix of the Kriking or generally in geostatistics, but also the global stiffness matrix \( K \) of the discretization of random fields (soil, mechanic, geotechnic etc. systems), when materials properties are described by means of random variables as Young modulus, Poisson ration etc.

\[
SFE = \sum_{j=0}^{\infty} K_j \varphi_j \quad \text{where} \quad K_j = E[K \varphi_j] - \int_{\Omega} B^T \cdot E[D(\theta) \varphi_j] \cdot Bd\Omega_e
\]

where:
- \( \xi = \xi_k(\theta), k=1,\ldots,M \) denotes the set of standard normal variables appearing in the discretization of all input random variables and \( \{\varphi(\xi)\} \) are multidimensional Hermite polynomials, \( B \) is the deterministic matrix that relates the component of the strain to the element model displacement and \( D(\theta) \) is the random elasticity matrix. In case of an isotropic elastic material with random independent Young’s modulus \( E \) and Poisson’s ratio \( \nu \) the elasticity matrix may be written

\[
D = E\left(\lambda(\nu)D_1 + 2\mu(\nu)D_2\right)
\]

where \( \lambda(\nu), \mu(\nu) \) are function of \( \nu \), which depend on the modeling \( D_1, D_2 \) are deterministic matrices.

Taking into account the our SFE algorithm is independent of integrand form, the proposed SFE could be applied in integrands that appear in geostatistic, SFE methods etc.

**SFE in reliability analysis.**

Structural reliability analysis of large scale mechanical, geotechnical, hydro technical etc. systems require the use of a finite element code coupled with a reliability method. The efficiency of the method in terms of number of calls to the finite element code is crucial when each single finite element run is computationally expensive. In this view we propose our approach as quadrature method in order to compute different objects of reliability analysis which include stochastic integrals. To simplify the idea let’s consider a mechanical (geotechnical) model in which we calculate the i-th- moment [7].
\[ E \left[ S^t(X_1, \ldots, X_N) \right] = \int_{\mathbb{R}^N} s(x_1, \ldots, x_N) \varphi(x_1) \ldots \varphi(x_N) \, dx_1 \ldots dx_N \]

where \( X = \{X_1, \ldots, X_N\} \) is a joint distribution, \( \varphi(.) \) is the PDF of a standard normal variable and \( s(x_1, \ldots, x_N) \) is usually known through a finite element code., \( w_k \) are integration weights. This moment can be estimated :

\[ E \left[ S^t(X_1, \ldots, X_N) \right] \approx \sum_{k_1=1}^{NP} \cdots \sum_{k_N=1}^{NP} w_{k_1} \ldots w_{k_N} \left[ s(x_{k_1}, \ldots, x_{k_N}) \right]^t \]

\( NP \) is the order of quadrature scheme. It is seen that \( NP^N \) evaluations of \( S \) are needed apriori. also there are an analogy between \( w_k \) and \( H_k \) of our SFE proposed when the space is \( \mathbb{R}^3 \) analog

**Conclusion**

The SFE proposed could be applied for different type of stochastic integrals (i.e. stochastic volume of fluids in porous media etc ), in geostatistic: covariances, moments, different type of krikings, elasticity theory : elasticity, stiffness, damping etc matrices , in reliability theory: discretisation forms , error estimators, second moment approaches, spectral stochastic methods ,stochastic differential equations, random spatial variability etc .In now days , SFE are indispensable in geotechnical plans, studies , designs , projects etc for reliability, risk analysis, stochastic optimization [14],[17],optimal decision making etc

**References**


