On the mechanics of "false vaults": new analytical and computational approaches

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ABSTRACT

The aim of this paper is to present new analytical and computational approaches for assessing the structural safety of "false vaults" structures like *Trulli*, and more generally for corbelled structures. In particular, the proposed procedure is capable of taking into account the three-dimensional behavior of such complex masonry structures.

Keywords: *masonry, corbelled vaults, historic constructions, equilibrium analysis, thrust network analysis*

1. INTRODUCTION

In the Mediterranean area, buildings covered by corbelled vaults made of purely horizontal stone layers, slightly cantilevered toward the center until meeting at the top, comprise a widespread and valuable heritage that deserves protection and enhancement.

Trulli are a remarkable example of vernacular corbelled dry-stone structures, typically located in Apulia, Italy. Whereas this kind of constructions appears to be the result of a self-building process, actually *Trulli* are the expression of a considerable constructive knowledge, handed down over time. Indeed, the architectural typology stems from the Mycenaean Tholoi built during the late Bronze Age (Greece XIV century B.C.).

The arrangement of *Trulli* vaults in horizontal courses of slightly cantilevered stone elements is obtained by using a very simple technique, that requires a minimum level of workmanship on the material and, at same time, do not need the installation of temporary supporting structures, generally employed during the building of ordinary domes.

Because their particular construction, "false vaults" like *Trulli* behave very differently from other kind of masonry vaults; nevertheless, and in spite of their extensive presence, this kind of masonry constructions have received a scarce interest from the scientific community, and consequently their mechanics is not completely understood yet.

Here, starting from a deep investigation on the building techniques of *Trulli* and on the employed materials, we underline the main mechanical issues of their structural behavior, and we propose an equilibrium approach, based on the Lower Bound theorem of Limit Analysis [11], aimed at assessing their structural safety under gravitational loads.

2. BUILDING TECHNIQUE AND STRUCTURAL BEHAVIOR

Trullo is generally a corbelled dry-stone building, covered usually with a pseudo-conical dome. It is generally composed by two main structural elements: the basement and the dome.

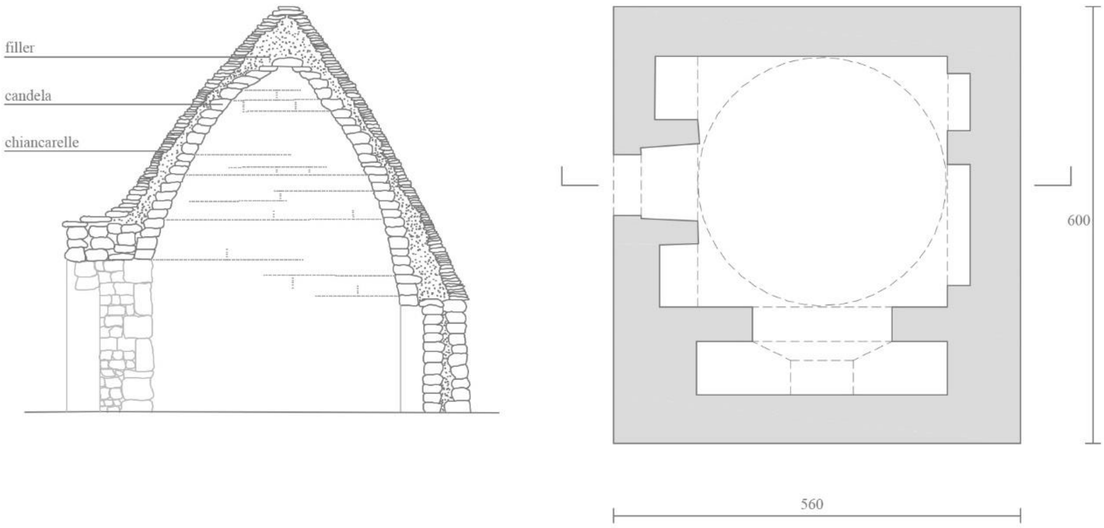

Fig. 1. Plan and section of case study Trullo in Alberobello

The basement, which can be circular or quadrangular, is made up of layers of large calcareous stones; the thickness of the walls is generally large: in most cases around *1* meter.

The dome usually has a pseudo-conical shape and it is composed of three main parts (Figure 1). The internal layer, generally with a constant thickness, is known as "candela" and it is built by laying subsequent separate jutting rings of stones, without usage of mortar. Special care is dedicated to achieve continuity of the ring brickwork, filling gaps between blocks through little pieces of stones. The finishing of the *trullo* dome is made by a coating of "chiancarelle", thin layers of limestone with a thickness of between 4-8 cm that are placed on the filling material with an outward slope in order to facilitate the water drainage. Only the first layer (candela) has a structural function, and consequently the other two layers can be regarded only as non-structural masses.

Structural behavior of corbelled vaults was addressed in a very limited literature; for example, since the 1980s some works were focused on the structural analysis of Mycenaean tholoi. In particular, in [7] an in-depth structural analysis on five Mycenaean tholoi is carried out using a multidisciplinary approach, connecting archeological data to the static framework of the so-called corbelling theory, which assumes that forces are transferred only vertically between pair of ashlars in contact. According to this hypothesis, the analysis of limit equilibrium configurations is performed by considering an infinitesimal meridian wedge of the dome, and by balancing the stabilizing moments and the overturning moments related to the overhanging masses. Thus, forces between stones acting along parallels, due to interlocking and friction, are not taken into account. Consequently, this approach is equivalent to regarding the structure as a set of independent corbelled meridian arches, without taking into account the three-dimensional behavior of the whole structure coming from the interactions of the meridian arches in the parallel transverse direction.

In [3], corbelling theory is applied to the study of corbelled arches, tholoi and *trulli*. The strong simplifications made by corbelling theory involve significant shortcoming in view of an accurate description of the actual structural behavior of corbelled constructions. The limitations of corbelling theory are emphasized in [8], where the cooperation among stones in the horizontal parallel rings is shown to be crucial for the interpretation of the mechanical behavior of corbelling domes.

Classical approaches for equilibrium analysis of curved masonry like the thrust line analysis, an effective method for determining the structural safety of two-dimensional masonry constructions do not apply for corbelled structures because the particular interactions of the ashlars. For example, in the case of *trulli* and by considering the dead load distribution for the dome (candela, filler and chiancarelle), it is not possible to find a thrust line contained within the thickness of the dome (Figure 2). As a consequence, the thrust line analysis leads to the wrong conclusion that this kind of structures should never be safe.

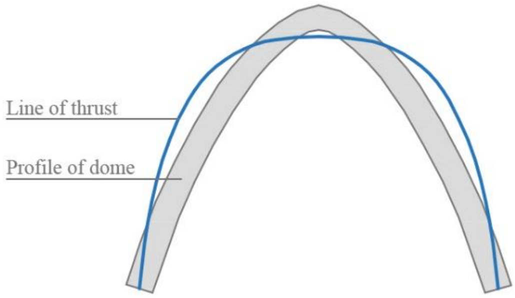

Fig. 2. Line of thrust and the structural profile of trullo

Actually, for the *trullo* case and more generally for dry stone constructions, horizontal rings of stones are capable to bear compression along parallels, and thus to resist to inward overturning forces. This capacity is fully deployed also when the construction of the ring is not completed thanks to the friction between ashlars lateral surfaces. A practical evidence of the effectiveness of this constructive technique is provided by several example of partially collapsed domes, showing that a meridian wedge is still able to stand without the collaboration of the collapsed surrounded structure (Figure 3).

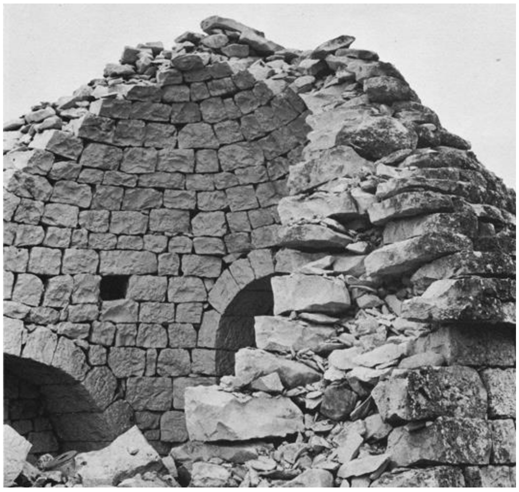

Fig. 3. Example of partial collapse of trullo vault [1]

Recently, Rovero & Tonietti [15] has proposed a modified corbelling theory for domes with horizontal layers based on balance equations that are capable to taking into account also the collaboration among adjacent infinitesimal wedges, due to the friction among blocks. This approach is then more accurate for describing the mechanics of corbelled vaults then the basic corbelling theory, but still to be improved in order to get accurate predictions of the actual structural behavior of these structures.

3. CASE STUDY

In view of the above considerations, we study the structural behavior of *trulli* vaults by applying two different approaches to a case study represented by the corbelled dome of a rural *Trullo* (Figure 4). In particular, the examined *Trullo* is located in Alberobello, a city of Valle d'Itria, in the middle of Apulia, UNESCO site since 1996.

Fig. 4. View of case study trullo in Alberobello

The dome is built on a well-clamped squared basement made of irregular stones, by progressively superposition of jutting rings of dry stones without using mortar. Ashlars are fairly irregular, but they are cut slantwise along the intrados to obtain a continuous surface at the inner side of the dome. It covers a span of 3.44 m, while the thickness of the structural layer (candela) could be considered approximately constant and equal to 0.26 m.

The first examined approach is based on the Thrust Network Analysis (TNA), a fully three-dimensional computational method that uses reciprocal force diagrams of graphic static to obtain compression-only spatial networks in equilibrium with the self-weight. The integration of the method with optimization algorithms allows for finding possible three-dimensional equilibrium compression spatial networks laying within the thickness of a masonry vault.

The second approach is based on the corbelling theory. First, we apply the standard corbelling theory proposed in [3]; then we apply a new formulation, which improves the modified corbelling theory suggested by [15]. In both cases analytical models were implemented in the software Wolfram Mathematica, used for finding close-form and numerical solutions of the differential equations governing the problem.

Assumptions for the purpose of analysis

Heyman [11] developed a Limit Analysis approach valid for masonry structures starting from the following constitutive assumptions on the masonry behavior (see also [9]):

(i) masonry is incapable of withstanding tensions;
(ii) masonry has infinite compressive strength;
(iii) masonry has infinite shear strength.

For the special kind of masonry structures here examined it is convenient for the purpose of analysis to add to the Heyman’s assumptions the following hypothesis:

(iv) no vertical joints in horizontal layers;
(v) no mortar and infinitesimal distance between overlapping layers.

Thrust network Analysis of a trullo dome

For the analysis presented in this Section, we adopt the implementation of the TNA method in RhinoVault [14], a funicular form finding plug-in of the architectural CAD parametric software Rhinoceros, aimed at the design of freeform thin shells. In order to extend the capability of RhinoVault to the search of equilibrium thrust networks in masonry-vaulted structures of given geometry, we used Python-scripts and an Evolutionary Optimization Algorithm implemented in Grasshopper, a visual programming language for Rhinoceros. In particular, the Optimization Algorithm allows for fitting the thrust network within the thickness of the masonry vault. The adopted procedure is characterized by a clear visualization of the results and by an easy and fast interaction with the analysis process.

The main steps of the Thrust Network Analysis (TNA) method of Block & Ochsendorf [5] are summarized in [10].

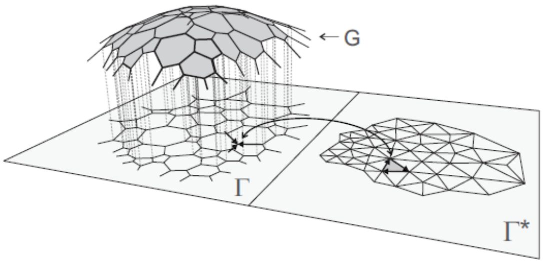

Fig. 5. Relationship between compression shell (G), its planar projection (primal grid Γ) and the reciprocal diagram (dual grid Γ*) to determine equilibrium [5].

The topology of the primal grid Γ used for the analysis has been obtained by the assumption that the forces be transferred not only along the vertical direction but exist a not negligible horizontal arch effect.

We analyze the vault for different loading conditions, specifically:

a) only the self-weight;
b) the self-weight and the weight of the infill;
c) the self-weight and a single eccentric point load.

When the optimization algorithm produces a result, the ratio between the actual vault thickness and the thickness of the thinnest possible vault geometry enveloping the funicular solution can be considered as a Geometrical Factor of Safety (GFS) [12], which synthetically gives a measure of the structural safety of the vault.

The used primal grid Γ, represented in Figure 6, has *203* edges and *73* nodes. By the application of the TNA method, thanks to manual manipulations of the dual force diagram Γ* and by using an Evolutionary Optimization Algorithm, it was possible to find lower-bound solutions for each of the above mentioned load cases in a relatively simple and fast way.

Tab. 1 summarizes the results in term of maximum and average vertical deviations of the node of thrust network from the middle surface, denoted as $|z - z^M|_{max}$ and $|z - z^M|_{av}$, respectively, and in term of GFS.

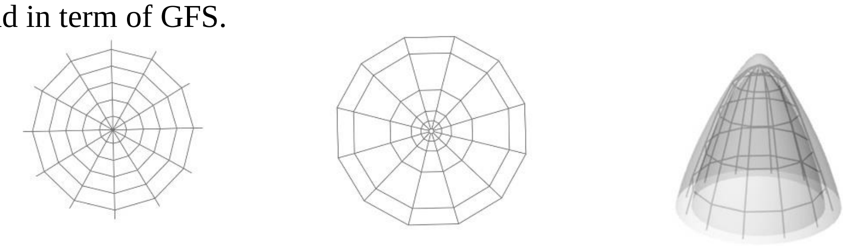

Fig. 6. Results of TNA analysis (a) Primal Grid Γ; (b) Dual force diagram Γ*; (c) 3D Thrust Network fit in the thickness of masonry vault.

Under the self-weight the thrust network fits the middle-surface remarkably well. The force diagrams clearly visualize the internal force distribution of the obtained solution (Figure 6). The addition of the load of the infill does not substantially vary the solution. The main vertical deviation occurs at the top of vaults is limited to the *0.2 %* of the span of the dome.

Load	Results		
	$\|z-z^M\|_{max}$ *[mm]*	$\|z-z^M\|_{av}$ *[mm]*	*GFS*
a	81	31	3.21
b	102	45	2.84
c	157	84	1.58

Tab 1. Results of the analysis with different loading condition.

Finally, the load condition including a single eccentric point load is a strategy for assessing the safety of the vault under general live loads. We analyzed the effect of a *5 kN* point load in various eccentric nodes of the network, and we obtained always solutions contained into the thickness of the masonry.

Structural Analysis by classical Corbelling Theory

The model proposed by Benvenuto & Corradi [3] study the limit equilibrium of a corbelled structure by considering as unknown the functions describing the shape of the intrados and the extrados. This way, a complex set of integral equations is obtained, and closed form solutions are possible only in special cases. In particular, with reference to Figure 7, and by considering an infinitesimal wedge of the dome, for any point P(x) belonging to the intrados curve *y(x)*, two region are defined: *PQR*, whose mass is related to the stabilizing moment (M_S) and *PROA*, whose mass generates the overturning moment (M_R). M_S and M_R are calculated with respect to a rotation axis passing through the point *P*.

Neglecting, as proposed by Bouguer [2], the contribution of the triangles like *P'Q'Q''*, M_R and M_S can be written as follows:

$$M_R = \gamma \int_0^x \left(y(\xi) - Y(\xi)\right)\left(x-\xi\right)\left(\xi d\varphi\right) d\xi \tag{1}$$

$$M_S = \frac{1}{6}\gamma\left(y(x) - Y(x)\right) h^2(x)\left(x d\varphi\right) \tag{2}$$

where *y(x)* an *Y(x)* are the ordinates of *P* and of a corresponding point *R* on the extrados; γ is the specific weight; and *h(x)* is the thickness of the dome at *x*.

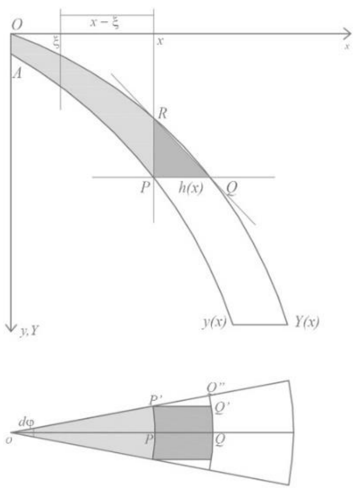

Fig. 7. Infinitesimal meridian wedge in plan and section which is analyzed with the corbelling theory.

The expression of the stabilizing moment M_S in (2) is approximated by replacing the extrados curve through the secant *RQ*. The thickness *h(x)* can be then express as follow:

$$h(x) = \frac{y(x) - Y(x)}{Y'(x)} \tag{3}$$

The balance equation can be written in the form:

$$M_R = \rho M_S \tag{4}$$

where $\rho \le 1$ is a suitable safety coefficient. Eq. (4) comprises a set of two equalities and three unknown function *y(x)*, *Y(x)* and *h(x)*. By making an assumption on *h(x)*, for example by setting $h(x)=h_0$ (constant), the analysis can determine the intrados and extrados shape functions *y(x)* and *Y(x)*.

The equilibrium equations (4) with $\rho=1$ can be reduced in differential form by differentiating two time with respect to *x*; indeed, we obtain:

$$\left(y(x)-Y(x)\right)-\frac{1}{6}h_0^3 Y'''(x)=0 \tag{5}$$

Boundary conditions can be express for *x=0*, i.e. at dome keystone, as:

$$Y(0)=0;\ \ y(0)=m;\ y'(0)=n \tag{6}$$

where *m* and *n* are parameters depending by geometry of the dome.

The system of differential equations (3) and (5) with the boundary condition (6) can be solved in closed-form obtaining as the solution the profile of intrados and extrados of the corbelling dome.

For the *trullo* under investigation, by applying the described approach we obtain the two limit curves of the intrados and the extrados, with *m=0.26* and *n=2/3*. These curves are plotted in Figure 8 in comparison with the real profiles of the intrados and the extrados. Figure 8 clearly shows a significant deviation between theoretical and real profiles. This suggests, as already underlined above, that a model based only on the overturning balance of infinitesimal dome wedges is not capable to describe the structural behavior of corbelled domes.

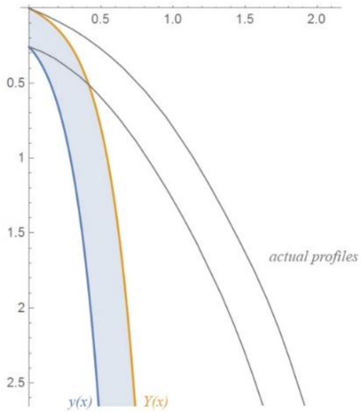

Fig. 8. Comparison between real and theoretical profiles obtained by corbelling theory

Structural Analysis by modified Corbelling Theory

In order to overcome the highlighted limitations of corbelling theory, Rovero & Tonietti [15] has recently developed a modified corbelling theory capable to taking into account the collaboration between meridian wedges due to the actions along parallels, that are induced by interlocking and friction between stone blocks.

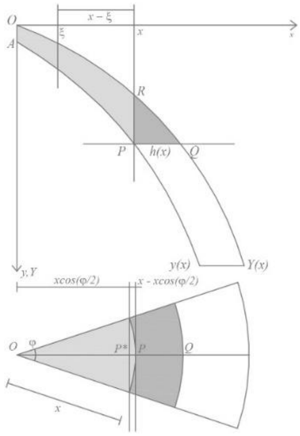

Fig. 9. Finite dimension meridian wedge in plan and section.

To do this, meridian wedges with finite dimensions are considered (their dimensions depend on a parameter φ representing the width of the meridian wedge), reproducing the continuity along parallels, and thus taking into account the transversal static cooperation among infinitesimal meridian wedges.

Since the assumption of finite-size meridian wedges the overturning axis moves toward the center of the dome from *P* to *P** (Figure 9), thereby increasing the stabilizing moment (M_S) with respect the overturning moment (M_R). The angular width φ of the meridian wedge conceptually expresses the minimal extension of the static cooperation along the horizontal rings necessary to achieve the limit equilibrium state.

In this formulation, the stabilizing moment (M_S) is still determined in the same approximate form as in [3] but, now, it is not possible to find a closed-form solution and the problem can be solved only numerically.

Here, we further improve the approach in [15] by proposing a new integral formulation of the problem. Again, it is not possible to find closed-form solutions but the new procedure here proposed allows for obtaining definitely less conservative results with respect to those obtainable by applying the procedure in [15].

With reference to Figure 9, for evaluating the expressions of the overturning moment M_R and of the stabilizing moment M_S the width φ determines the distances from the rotation axis of the centroids of the infinitesimal slices subdividing the region *PROA* and of the infinitesimal slices subdividing the region *PQR*:

$$M_R = \gamma \int_0^x \left(y(\xi) - Y(\xi) \right) \left(x \cos\frac{\varphi}{2} - \xi \right) \left(\xi d\varphi \right) d\xi \tag{7}$$

$$M_s = \gamma \int_x^{x+h_0} \left(y(x) - Y(\xi) \right) \left(\xi - x \cos\frac{\varphi}{2} \right) \left(\xi \varphi \right) d\xi \tag{8}$$

According to the above, by assuming constant thickness $h=h_0$ and $\rho=1$, the equilibrium equation now takes the form:

$$\int_0^x \left(y(\xi) - Y(\xi) \right) \left(x \cos\frac{\varphi}{2} - \xi \right) \left(\xi \varphi \right) d\xi - \int_x^{x+h_0} \left(y(x) - Y(\xi) \right) \left(\xi - x \cos\frac{\varphi}{2} \right) \left(\xi \varphi \right) d\xi = 0 \tag{9}$$

By differentiating two time with respect to *x*, Eq. (9) can be reduced in the following differential form:

$$-\frac{1}{6}h_0\varphi\left\{\begin{array}{l}-6\left[-h_0-2x+(h_0+3x)\cos\frac{\varphi}{2}\right]y'(x)+\\ \left[2(h_0^2+3h_0x+3x^2)-3x(h_0+2x)\cos\frac{\varphi}{2}\right]y''(x)\end{array}\right\}=0 \quad (10)$$

The system of differential equations (3) and (10) with the boundary condition (6) can be solved numerically for obtaining as solutions the shape functions *y(x)* and *Y(x)* of the intrados and the extrados of the vault, respectively. Differently from the basic model in [3], now the solutions of the problem depends on the parameter φ.

In Figure 10 we have compared the actual profiles of the vault with the theoretical limit curves *y(x)* and *Y(x)* obtained by solving (10). Since the solutions depend on the angle φ, we choose the value of φ which minimizes the maximum horizontal distance between the actual axis of the dome and the theoretical profile, i.e., $\varphi=60.8°$ for the case study. Notice that our procedure allows for obtaining a theoretical profile closer to the actual profile of the vault than the theoretical profile obtainable by the procedure in [15], which already substantially improves the results of the approach in [3].

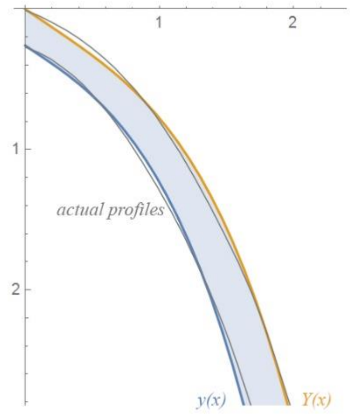

Fig. 10. Actual and theoretical profiles obtained by new formulation of modified corbelling theory.

Fig. 11.

CONCLUSION

The analysis of an Alberobello *trullo* highlighted the fully three-dimensional behavior of these particular masonry corbelled vaults, and the shortcomings of classical approaches as the thrust-line analysis and the basic corbelled theory. The Thrust Network Analysis (TNA), a three-dimensional computational method, is applied to obtain lower-bound solutions for the corbelled vault and for determining the structural safety under vertical loads by evaluating a reliable lower-bound of the Geometrical Factor of Safety (GFS). The analysis is relatively fast and easy, making the TNA method appropriate for preliminary studies and useful for practical applications, when more complex and more time-demanding non-linear finite element analysis or discrete element analysis are not worthwhile.

Furthermore, we proposed a new formulation by improving a recently proposed analytical model based on the corbelling theory, and taking into account the structural collaboration between adjacent meridian wedges due to the friction among blocks and layers of stones. The stabilizing effect of this collaboration is quantified by an angle φ that identifies the finite width of the meridian wedge necessary to guarantee the equilibrium to the overturning. The angle φ

can be assumed as a measure of the efficiency of the dome shape under uniform vertical loads: smaller angles can be associated the more efficient shapes of the profile of the vault. In this vein, the new integral formulation here proposed allows to obtain less conservative results than those in [15].

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