QUADRUPLETS

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SABAH BUSHAJ

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Sabah Bushaj

ABSTRACT

QUADRUPLETS

Bushaj, Sabah

M.Sc., Department of Computer Engineering

Supervisor: Dr. Arban Uka

In this paper we treat the one of the most known and popular topics. In 1963, Barning discovered a ternary tree with nodes the Primitive Pythagorean triplets by using 3 matrices that helped him propagate through the tree. Independently, Hall found the same thing after seven years.

In this paper we go through the ways that we know to generate triplets and try them to help us find another tree, this time a little bit different based on a new conjecture. A new way to find triplets also is found which we programmed to find quadruplets but with slight modification of the code can also find the matrices found by Barning and Price. We also try another method using matrix transformations. We use matrices even though they are inefficient from computation time point of view but they are an elegant representation and useful to observe patters.

During our work with the topic, interesting results are found related to the tree of Barning and Price which we have stated them at the results.

Promising results are also found related to the quadruplets tree. Some matrices that help us propagate 3 steps are found and give an insight of how the tree may be formed.

Keywords: Primitive triplets, Barning, Price, matrix.

ABSTRAKT

KUADRUPLETET

Bushaj, Sabah
Master Shkencor, Departamenti i Inxhinierisë Kompjuterike
Udhëheqësi: Dr. Arban Uka

Ne kete studim ne do trajtojme nje nga temat me popullore dhe nder me te njohurat. Ne 1963, Barning zbuloi nje peme pjesetare te se ciles ishin te gjithe tripletet Primitive te Pitagores duke perdorur 3 matrica te cilat e ndihmonin ate te shkonte sa ne nje triplet ne nje tjeter. Ne menyre te pavarur, Hall gjeti po ashtu te njejten peme pas 7 vitesh.

Ne kete studim ne merremi me menyrat e deritanishme te gjenerimit te tripletve dhe se si mund ti perdorim ato per te na ndihmuar qe te gjejme nje peme tjeter, kete here te bazuar ne nje supozim (ekuacion) tjeter. Eshte zbuluar nje menyre e re e gjetjes se tripletave te cilen e perdorim poashtu edhe per te gjetur kuadrupleta por me nje modifikim te vogel te kodit ne mund te gjejme edhe matricat e Barning dhe Price.

Ne poashtu provojme nje menyre tjeter duke perdorur transformime te matricave. Ne perdorim matrica edhe pse jane joeficiente persa i perket kohes per llogaritje por jane nje prezantim elegant dhe shum me te lehta per te vezhguar modele.

Gjate punes tone, kemi gjetur rezultate interesante gjithashtu ne lidhje me pemen e Barning dhe Price te cilat perfshihen tek rezultatet.

Rezultate premtuese jane gjetur poashtu ne lidhje me pemen e kuadrupleteve. Disa matrica qe na ndihmojne qe te zbulojme deri ne 3 hapa te pemes jane gjetur dhe japin nje ide se si mund te formohet pema e kuadrupleteve.

Fjalët kyçe: Triplete primitive, Barning, Price, matrica.

Dedicated to my family!

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LIST OF ABBREVIATIONS

FLT	Fermat’s Last Theorem
PTT	Primitive Triplets Tree
QDP	Quadruplet
PPT	Primitive Pythagorean Triple
EEC	Extended Euler Conjecture

CHAPTER 1

INTRODUCTION

Even though is a short life, we do know that a lot of people cannot be considered dead. And that due to the contributions they have provided to humanity. Some great people, no matter how short their life has been they left the world with a lot of miracles. We do not know a lot about the life of Diophantus, at least not as much as we would like to know, but what he left to the world was the start of a long-lasting miracle. Sometimes he is called as the “the father of algebra” which is somehow fair due to the great work he left when he wrote the book series known as Arithmetica (many of them lost today).

When it comes to his life, we know very little. We know he lived in Alexandria around AD 200 and 214 to 284 or 298. What we know for him is from a famous grammarian and mathematician from the 5^{th} century Greece, mostly written as poems or pieces of writing in the form of number games and puzzles.

When it comes to the reason we still know him even after thousands of years, we are unlucky that most of the work he did in the book series Arithmetica is lost. Even with what was left (6 books) a mathematical revolution started. Arithmetica is a collection of problems giving numerical solutions to determinate and indeterminate equations. Of the thirteen editions he published (at least belived so) 6 are found and translated, and some believe that the books found in 1968 in Arabic are from Diophantus.

Arithmetica was not the only book written by Diophantus. In Arithmetica he refers some of the theorems from another book of his, known as The Porisms (or Porismata). One of them is the lemma that states the difference between two rational cube numbers can be expressed as the sum of two other rational cube numbers.

Ex: given any a and b with $a > b$, there exist c and d, positive and rational such that

$$a^3 - b^3 = c^3 + d^3$$

Diophantus has a great contribution in different fields in arithmetic's.

Diophantus made a lot of advances in mathematical notation. He was the first person to use notations and symbolism to represent the equations. Before him, everyone would write the whole equations completely. Despite the advances he made, he still lacked some important notations in order to advance more in the equation representation. He only had a notation for only one unknown and also lacked *n* to express the general number.

The Diophantine Analysis is the study of seeking a solution for an integer solution to different equations. Whereas Diophantine Equations are polynomial equations but only seeking a positive integer solution. Most of the Diophantine equations lead to quadratic equations. Diophantine in his book studied different types of quadratic equations while we nowadays have only one type of quadratic equation.

Exa: $\mathrm{ax}^2 + \mathrm{bx} = \mathrm{c}$

That is because Diophantus did not have a notion for 0 and avoided negative coefficients by using only a, b and c positive numbers.

He called the equations leading to negative values as "meaningless" and "useless".

Exa: $4 = 4\mathrm{x} + 20$ is called by him as "absurd" and "useless" because it would lead to negative values of x. A positive solution was all he was looking for. There is no proof that Diophantus ever wondered or understood that the quadratic equations may have more than one solution.

Despite the differences that Diophantus had with the later mathematicians one thing is for sure: he is the person of finding the base of the arithmetics thus inspiring many other people obsessed with numbers.

1.1. Beauty of numbers

We are used to seeing numbers everyday at our daily routines. Reading newspapers or magazines, at work or in any other social activity. Everyday we have our experiences with numbers but we never stop and think about their beauty. It’s like walking through a garden and ignoring the smell of the roses.

There are two types of number properties, (i) those that are special cases, happening on only a type of number system and (2) those who are true for every number system.

a. Example 1

142 857 * 2 = 285 714

142 857 * 3 = 428 571

142 857 * 4 = 571 428

142 857 * 5 = 714 285

142 857 * 6 = 857 142

142 857 * 7 = 999 999

142 857 * 8 = 1.142 856

142 857 * 9 = 1.285 713

We can observe the symmetry between the number and the product only changes the order of the digits. Most suprising result is when the number is multiplied by 7, but what is more strange is that when multiplied by 8 and 9 if we removed the millions digit and add it to the units’ digit, the original number is formed.

b. Example 2

1 089 * 1 = 1 089

1 089 * 2 = 2 178

1 089 * 3 = 3 267

1 089 * 4 = 4 356

1 089 * 5 = 5 445

1 089 * 6 = 6 534

1 089 * 7 = 7 623

1 089 * 8 = 8 712

1 089 * 9 = 9 801

If you can notice, there's a pattern between the products. By looking at the 1st and the 9th product we can see that they are reverses of each other. And this pattern continues until the 5th product is a reverse itself, know also as palindromic number. And if even that is not enough also the number 10,989 * 9 = 98, 901. And goes on with 109,989 * 9 = 989,901 and so on.

Moreover, there is only one other number with 4 or <4 digits that has this property. The number 2,178 * 4 = 8,712 and it gets more interesting when we notice that 2,178 = 1,089 * 2.

Unlike other unusual oddities this is not depended on the numerical system but in operations.

We will provide one more example to show how intriguing can numbers be. This is a well known conjecture and a very old one that have been concerning mathematicians since 1930.

This is not a trick, it's not magic either, but no one knows why this happens.

It begins with a arbitrarly selected number, then

> If the number is odd, then multiply by 3 and add 1.
> If the number is even, then divide by 2.

Let's try it for the arbitrarily selected number 5:

5 is odd; therefore, we multiply by 3 and add 1 to get 16.

16 is even, so we divide by 2 to get 8.

8 is even as well, so we divide again by 2 to get 4.

4 is even too, again we divide by 2 to get 2.

2 is even, divide by 2 and we get 1.

Even though there's no general proof, it is believed that always the result is 1, despite the arbitrarily selected number at the beginning, here we started with 5. This is amazing! You can try it yourself by starting with another number and you will get the same result as well.

Despite monetary rewards being offered there still exists no proof for this conjecture. This problem is named as 3n+1 problem and has been proved that it works for every number until 1018.

c. Example 3: Friendly Numbers

Two numbers are said to be friendly if the sum of the divisors of one equals the second and the sum of the divisors of the second number equals the first number.

1st pair: 220 and 284

The proper divisors of 220 are 1 2 4 5 10 11 20 22 44 55, and 110.

Their sum is 1+2+4+5+10+11+20+22+44+55+110 = 284.

The proper divisors of 284 are 1, 2, 4, 71, and 142, and their sum is 1 + 2 + 4 + 71 + 142 = 220.

2nd pair: 17,296 and 18,416 discovered by Pierre de Fermat

The sum of the proper factors of 17,296 is

1+2+4+8+16+23+46+47+92+94+184+188+368 + 376 + 752 + 1 081 + 2 162 + 4 324 + 8 648 = 18 416

The sum of the proper factors of 18,416 is

1 + 2 + 4 + 8 + 16 + 1 151 + 2 302 + 4 604 + 9 208 = 17 296

1.2. Fermat Theorem

Fermat as a lawyer, not primarily a mathematician, contributed largely in science just like Joule who worked on a brewery, similarly to Faraday who worked as an assistant to a bookseller and bookbinder.

Fermat, despite his passion, mathematics, he had a normal everyday life where he worked as a lawyer and he was successful at that profession too. But what we are interested through this paper is his work he did in mathematics.

When Fermat moved to Bordeaux he was in contact with a lot of mathematicians and he was influenced a lot by their works, especially from Francois Vieta. Vieta was known for his work on new algebra because of his innovative idea to use letters as parameters into equations. He communicated frequently with his friends about his work, but they were mostly theorems, often with no proof at all. Secrecy was very common in the European society at that time (Wiles worked in secret too). This led to dissagreements between Fermat and his contemporaries.

Fermat, together with Rene Descartes, was one of the best mathematicians of the first half of 17th century. Bernstein in his book written in 1996 describe Fermat as follows: “Fermat was a mathematician of rare power. He was an independent inventor of analytic geometry, he contributed to the early development of calculus, he did research on the weight of the earth, and he worked on light refraction and optics. In the course of what turned out to be an extended correspondence with Pascal, he made a significant contribution to the theory of probability. But Fermat's crowning achievement was in the theory of numbers”.

One of the closest persons that Fermat communicated most was Carcavi. Fermat met Carcavi in Toulouse, despite sharing the same profession they also were astonished with mathematics. Fermat told Carcavi about the discoveries that he had made and he was really impressed by Fermat’s work. It was Carcavi that told Mersenne about the discoveries that Fermat had made regarding free falling bodies and he contacted Fermat himself. Fermat wrote back explaining descriptions and discoveries and how he believed that Galileo made some errors on the description of free falling. It is somehow interesting how Fermat had his first contact with the scientific community for a physical topic, since he was not very interested in the physical applications of mathematics.

Fermat's reputation grew very quickly when he sent Mersenne a letter containing also two problems on maxima and asked him to distribute this with the mathematicians at the scientific community.

Mathematicians found Fermat's problem very difficult and usually they could not be solved using current techniques, so they asked him to write the methods he used to solve them and he sent *Method for determining Maxima and Minima and Tangents to Curved Lines*, his restored text *of Apollonius's Plane loci* and his algebraic approach to geometry *Introduction to Plane and Solid Loci* to the Paris mathematicians.

Although his reputation was high, he failed to publish his work because he never was really interested into taking credit about his work. In the first letter to Mersenne he also writes:

> *I have also found many sorts of analyses for diverse problems, numerical as well as geometrical, for the solution of which Viète's analysis could not have sufficed. I will share all of this with you whenever you wish and do so without any ambition, from which I am more exempt and more distant than any man in the world*.

Later on he would get in a controversy with some of the Parisian mathematians for not agreeing on the same opinion. His reputation was a bit damaged because Parisian mathematicians had a lot of influence on the crowd and over the scientific community.

From 1643 to 1654 is a period when Fermat did not keep in touch with the Parisian mathematicians. It can be understood that there are different reasons for this. There was too much pressure from work which did not let much time to spend on mathematics. Secondly there was a civil war which affected Toulouse a lot. And lastly, from 1651 a plague that hit

Toulouse had great consequences, even on Fermat himself. Surprisingly, this period of time is considered the time that Fermat worked on number theory.

Fermat, is mostly known for his work on the number theory, especially the Fermat's Last Theorem which states that:

$$x^n + y^n = z^n$$

has no non-zero solutions for x, y and z when n > 2.

The most intriguing part about the theorem is what Fermat wrote on the margin of the Bachet's translation of Diophantus's *Arithmetica:*

I have discovered a truly remarkable proof which this margin is too small to contain.

These notes became famous only after Fermat died. It was his son, Samuel, who published them in 1670.

Many attenpts made for over than 300 years to prove FLT lead to many other discoveries, but only in 1995 would Andrew Wiles publish a proof to the theorem.

Still, Wiles proof still left us with a big question.

Did Fermat really have a proof about his theorem?

Tools and methods used by Wiles are way beyond the mathematics of Fermat, which questions the possibility that Fermat had a proof. But it's also the possibility that Fermat realized he made an error but did not correct his notes, which seems unlikely because he was precise in labeling his theorems and conjectures. This also raises the question, if he had a proof, why did he never told anyone about it?

Furthermore, the possibility of Fermat having a proof raises another big problem. Is there a way to simplify Wiles proof? Can we find a proof using less sophisticated tools and methods?

Personally, I do believe that Fermat had a proof to his theorem. (How do you argue on this?)

First of all, he is really precise when it comes to his notes and conjectures which rules out the possibility of him not correcting the comment.

Secondly, Fermat does not seem the type that would brag about his findings, he sees profit only in the satisfaction he gets from his hobby and him not telling anyone does not mean he was wrong, but maybe he just underestimated this as a problem and to him mattered more the other topics included in the papers exchanged with the other mathematicians of that time.

Lastly, Fermat was always a step ahead of the other mathematician's work, he would try them by sending them problems he had already solved. And I think if Fermat found something that challenged him, he would do more than just pass over it and not talk about it with anyone.

1.2.1. The Euler Conjecture

During our work though the thesis, we decided to modify the FLT since it was proven from Wiles. We thought that when increasing power, we should also increase the dimensions. So, $a^3 + b^3 + c^3 = d^3$ seems to be the logical step after $a^2 + b^2 = c^2$.

Unknown to us it had been that this was conjured by Euler since 1769, stated as for all integers n and k greater than 1, if the sum of n kth powers of non-zero integers is itself a kth power, then n is greater or equal than k.

In symbols,

$$\sum_{i=1}^{n} a_i^k = b^k$$

where n>1 and $a_1, a_2, \ldots, a_n$, b are non-zero integers, then n>=k.

Although Euler only proved that this is true for n = 3. Later on it was disproved for the cases of n = 4, 5, 7 and 8.

For the case of n = 3 (which we are interested in) Euler had provided a complete solution for positive and negative rational integers and they are given by equations:

$$a = (1-(p-3q) * (p^2+3q^2)) * r$$

$$b = ((p+3q) * (p^2+3q^2)-1) * r$$

$$c = ((p^2+3q^2)^2-(p+3q)) * r$$

$$d = ((p^2+3q^2)^2-(p-3q)) * r$$

Later on, we will use an approach based on these equations to try to find a generalized result that can give us a matrix to traverse through a tree of quadruplets.

1.3. A tree of n-tuples?

An n-tuple is a series of numbers that has n numbers. The n-tuples that we are going to consider here have a special property. The numbers in an (n+1) tuplet are such that the volume in n-dimendional space with a side equal to the n+1th number can be written as a sum of n-volumes in the n-dimensional space that have sides equal to the first n numbers in the tuplet.

$(a_1, a_2, a_3, \ldots a_n, b) \rightarrow a_1n+a_2n+a_3n+\ldots..a_nn=bn.$

1.3.1. Methods used to generate Triplets

There are many different methods used to generate triplets. We are going to describe briefly the methods that we can generate all the triplets [a, b, c] satisfying $a^2 + b^2 = c^2$

1.3.1.1. Euclid's Formula

Euclid's formula is the most fundamental formula for generating triplets using integers *m* and *n* where *m*>*n*. But in order for this to be a primitive triplet it's a must that *m* and *n* are coprime and that *m – n* is odd. If *m* and *n* are both odd the formula will work again but *a*, *b* and *c* with be even numbers that is we will have to dive by greatest common factor to get the primitive triplet. (Mitchell 2001)

The formula states that

$$a = m^2 - n^2,\ b = 2*m*n,\ c = m^2 + n^2$$

and thus *a*, *b* and *c* form a triplet.

Every primitive triplet is formed from a uniques pair of coprime m and n where one of them is even.

m	n	a	b	c
2	**1**	3	4	5
4	**1**	15	8	17
6	**1**	35	12	37
8	**1**	63	16	65
3	**2**	5	12	13
5	**2**	21	20	29
7	**2**	45	28	53
9	**2**	77	36	85
4	**3**	7	24	25
8	**3**	55	48	73
10	**3**	91	60	109

Table 1. List of triplets formed using m and n

It is interesting that, supposing a Pythagorean triangle has sides $m^2 - n^2$, $2 * m * n$, $m^2 + n^2$ (a, b and c), and the angle that is between the side $m^2 - n^2$ and the hypothenuse $m^2 + n^2$ ∂. We would have $\tan\partial = \frac{2mn}{m^2-n^2}$ and $\tan\frac{\partial}{2} = \frac{n}{m}$. (Malcolm Scott MacKenzie 1993)

1.3.1.2. Fibonacci's Method

The generation of Primitive Pythagorean triples using Fibonacci method is another old and useful way. In his book, The Book of Squares, (Liber Quadratorum in Latin) published in 1225, Leonardo of Pisa (aka Leonardo Fibonacci) anticipated a lot of the works of Fermat and Euler who would make a lot of advances later. It this book examines different topics of number theory; among them he presents an interesting way for finding Pythagorean triplets using odd numbers.

The basis of this method is that he uses the sequence of consecutive odd numbers 1,3,5,7,9… and the fact that the sum of the first *n* numbers is equal to n^2. If *m* is the *n*-th number in this sequence, then $n = \frac{m+1}{2}$.

The method is straightforward: we choose an odd square number in this sequence and let its position be n (ex: $m = 9 \Rightarrow 3^2 = a^2$ and $n = 5$). Then let b^2 be sum of the *n-1* terms of this sequence ($1 + 3 + 5 + 7 = 16 = 4^2 \Rightarrow b = 4$). And let c^2 be the sum of all the n terms ($1 + 3 + 5 + 7 + 9= 25 = 5^2 \Rightarrow c = 5$). In this way we have generated a primitive triplet [3, 4, 5].

1.3.1.3. Progressions of whole and fractional numbers

This is another old method for calculating triplets. This method was published in 1544 by Michael Stifel, a German monk and a mathematician. It is one of the earliest and one of the most important book in Germany, best-known book of the sixteenth century.

It takes into consideration the progression of whole and fractional numbers: $1\frac{1}{3}, 2\frac{2}{5}, 3\frac{3}{7}, 4\frac{4}{9}$ ...

This progression must satisfy 3 properties:

a. The whole numbers of the common series have same unity difference

b. The numerators of the fractions, added to the series are also natural numbers

c. The denominators of the fractions are the numbers of the series 3, 5, 7, 9, …

In order to calculate a Pythagorean triplet, select any term of the sequence and turn it into an improper fraction. For example, lets select the term $2\frac{2}{5}$ and turn it into $\frac{12}{5}$. The numerator and the denominator are the sides a and b while the 3rd side is found by adding 1 to the longest side (in this case 12+1 = 13). In this case, we generated the triplet [5,12,13].

Stifel's method was republished in 1694 by Jacques Ozanam where he also added another similar sequence that finds also primitive triplets.

1.3.1.4. Dickson's Method

Dickson is mostly known in Number Theory for his three-volume published book, History of the Theory of Numbers. This book is still refered to for many informations related to the topics of the volumes. It is simply a collection summarizing work done in number theory until 1920s, where Dickson list results and discussions of the various authors that have contributed before and then adds further disscusions.

Concerning our topic, Dickson has a very simple approach to find triplets. He states that in order to find triplets satisfying $a^2 + b^2 = c^2$, we need to find positive integers x, y and z such that $x^2 = 2*y*z$.

So, then we have:

$a = x + y, b = x + z, c = x + y + z.$

From this method all the Pythagorean triplets can be obtained. The Pythagorean triplets generated will be primitive only if *y* and *z* are coprime

1.3.1.5. Generation by use of Matrices and Linear Transformations

1.3.1.5.1. Barning Method

The Barning method goes a lot more further than just finding a way to generate triplets. Barning discorvered 3 matrices that whenever multiplied with a primitive triplet it will produce another primitive triplet, thus building a tree of triplets with the root as [3, 4, 5]. Every triplet produces 3 other triplets, this way building a ternary tree descending from the root.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

By multiplying one of these matrices with a triplet on the tree (starting from [3, 4, 5]) the following infinite ternary tree is formed.

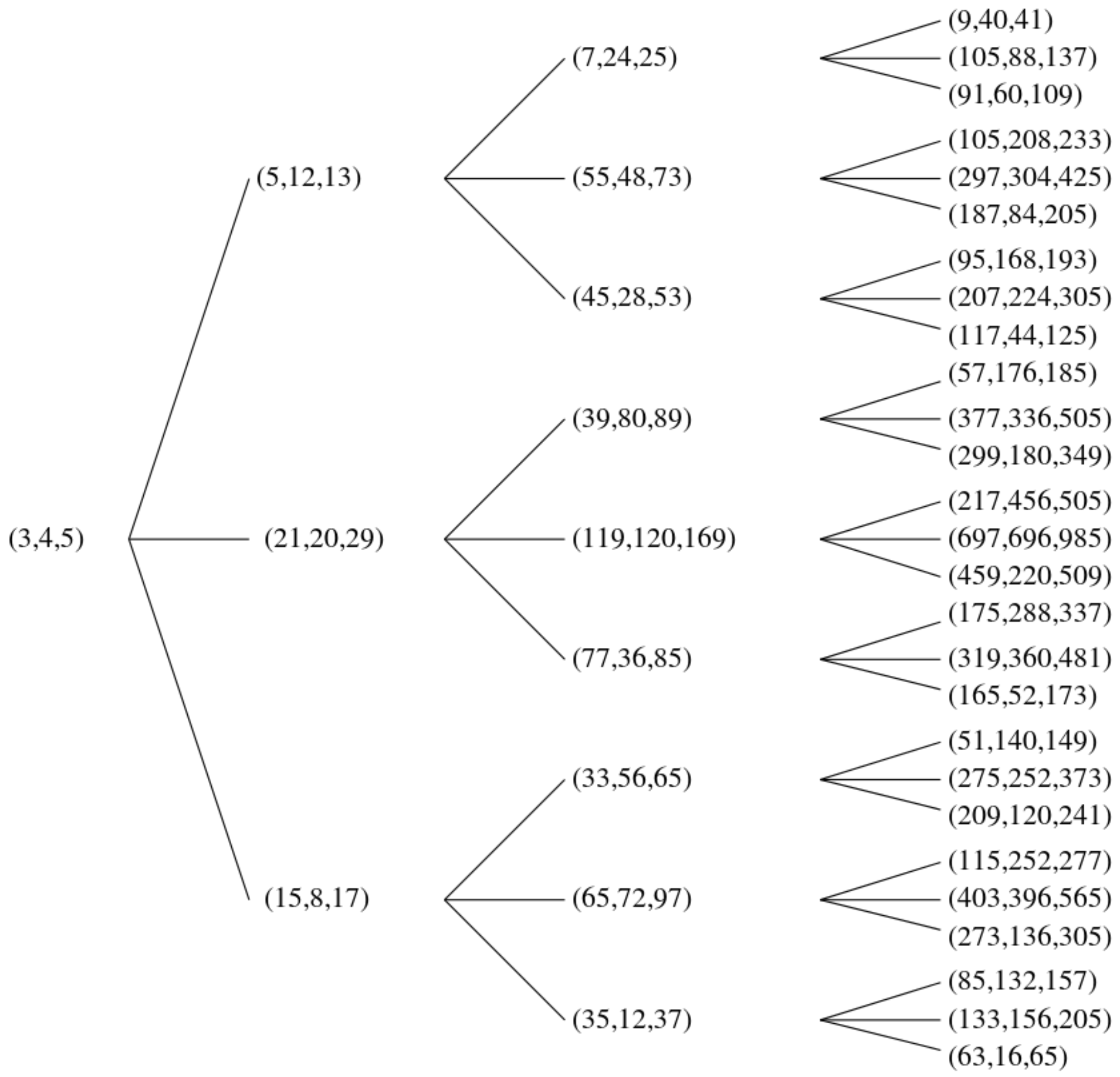

Figure 1. Barning Triplets Tree

What is most interesting is the way Barning got to this conclusion.

Let $a^2 + b^2 = c^2$ and $x^2 + y^2 = z^2$, we create a system of equations using the similarities between two primitive triplets we know: [3, 4, 5] and [15, 8, 17].

Let a = 3, b = 4, c = 5 and x = 15, y = 8, z = 17. They both satisfy the equations above and we know they are primitive triplets since they have no common divisor bigger than 1.

$$\begin{cases} x - a = 15 - 3 = 12 = t \\ y + b = 8 + 4 = 12 = t \\ z - c = 17 - 5 = 12 = t \end{cases} \Rightarrow \begin{cases} x = t + a \\ y = t - b \\ z = t + c \end{cases} \Rightarrow (t + a)^2 + (t - b)^2 = (t + c)^2$$

As a result, we get $t = -2a + 2b + 2c$ using the system of equations above,

$$\begin{cases} x = t + a => t + a = -2a + 2b + 2c + a \\ y = t - b => t - b = -2a + 2b + 2c - b \\ z = t + c => t + c = -2a + 2b + 2c = c \end{cases} \Rightarrow \begin{cases} -a + 2b + 2c \\ -2a + b + 2c \\ -2a + 2b + 3c \end{cases} \Rightarrow \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

Comparing the matrix to the matrices A, B and C we conclude that using similarities between the triplets [3, 4, 5] and [15, 8, 17] we have found matrix C. So, matrix C propagates us from [3, 4, 5] to [15, 8, 17] and it will continue through [35, 12, 37] and so on.

Similar to this method, three other matrices found by Price that produces also all primitive triplets, but in a different order than the set of matrices found by Barning.

$$A' = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \qquad B' = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix} \qquad C' = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Both of methods are accurate, just the set of children for different parents differ. For example, using the parent [3, 4, 5] methods generate different results, both true.

	[3, 4, 5]	
A	B	C
[5, 12, 13]	[21, 20, 29]	[15, 8, 17]

	[3, 4, 5]	
A’	B’	C’
[5, 12, 13]	[15, 8, 17]	[7, 24, 25]

Figure 2. Difference between Barning and Price Triplets

1.4. Aim and Novelty of the Study (Quadruplets)

The goal of this study was to study if by increasing the dimensions of the FLT equation we would be able to make a generalization and solve it. But unknown had been to us that a very similar idea was in fact proposed by Euler earlier known as Extended Euler Conjecture (EEC) which is a very similar conjecture but its aim is not the same as our conjeture. Together with this we have found a new way of generating triplets and also some part of the quadruplets tree. We aim to find the matrices (4x4 in this case) that will make possible for us to jump from one quadruplet to another one by saving the properties we need as in the case of PTT.

CHAPTER 2

PRESENTATION OF OUR APPROACH

2.1. Matrix representation

In our work we will try to pay importance to the representation of the relations using linear transformations, or using matrices. Matrices in their essence express a transformation, or express a conservation of a certain feature, or a conservation of a certain relationship between series of numbers. They are a new mathematical construct developed in the early 20th century as it was found to be very useful when explaining quantum mechanics, and the associated terms are mainly German (exa: eigenvalue). In terms of computational time matrices are not efficient, but we will use them because of the elegant representation and the useful concepts related to spaces that can be derived once we observe a pattern.

One approach is to use the equations that we have for the [a, b, c, d] as a function of [p, q] that Euler has used to find all the solutions to quadruplets. The smallest examples to this form ($a^3 + b^3 + c^3 = d^3$) are $3^3 + 4^3 + 5^3 = 6^3$

$$1^3 + 6^3 + 8^3 = 9^3$$

$$7^3 + 14^3 + 17^3 = 20^3$$

as equation form 1, and

$$1^3 + 12^3 = 9^3 + 10^3$$

$$2^3 + 16^3 = 9^3 + 15^3$$

as equation form 2. While we know that the equation $1^3 + 12^3 = 9^3 + 10^3 = 1729$ is taxicab(2) found by Ramanujan. We will choose $a^3 + b^3 + c^3 = d^3$ (eq.1) as the form to solve because Euler has *completely* solved this for positive or negative rational solutions and it is given by,

$$a = (1 - (p - 3q)(p^2 + 3q^2))r$$

$$b = ((p + 3q)(p^2 + 3q^2) - 1)r$$

$$c = ((p^2 + 3q^2)^2 - (p + 3q))r$$

$$s = ((p^2 + 3q^2)^2 - (p - 3q))r$$

where the variable *r* is a scaling factor reflecting the equation's homogeneity. In other words, r plays the role of GCD and makes possible to get the primitive quadruplet.

p	q	a'	b'	c'	d'	r / gcd	a	b	c	d
1	1	9	15	12	18	3	3	5	4	6
2	1	8	34	44	50	2	4	17	22	25
2	2	65	127	248	260	1	65	127	248	260
1	2	66	90	162	174	6	11	15	27	29
3	1	1	71	138	144	1	1	71	138	144
1	3	225	279	774	792	9	25	31	86	88
3	2	64	188	432	444	4	16	47	108	111
2	3	218	340	950	968	2	109	170	475	484
3	3	217	431	1284	1302	1	217	431	1284	1302

Table 2. List of quadruples formed using p and q

So

$$\frac{a}{r} = -\left\{ [p \quad q] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \right\} \left\{ [1 \quad -3] \begin{bmatrix} p \\ q \end{bmatrix} \right\} + 1$$

$$\frac{b}{r} = \left\{ [p \quad q] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \right\} \left\{ [1 \quad 3] \begin{bmatrix} p \\ q \end{bmatrix} \right\} - 1$$

$$\frac{c}{r} = \left\{ p \quad q \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{matrix} p \\ q \end{matrix} \right\} 2 - \left\{ [1 \quad 3] \begin{bmatrix} p \\ q \end{bmatrix} \right\}$$

$$\frac{d}{r} = \left\{ p \quad q \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{matrix} p \\ q \end{matrix} \right\} 2 - \left\{ [1 \quad -3] \begin{bmatrix} p \\ q \end{bmatrix} \right\}$$

Thus we tried to find a matrix that is between the different p and q values in a form or a series, matrix that can give us (p_1, q_1) by knowing what (p_2, q_2) is.

This takes into consideration the possibility that the solution may not just be a matrix but it also may be an equation of matrices that saves different properties of the quadruplets from one to another which is not strange because from triplets to quadruplets the complexity increases.

So, finally we have

$$\begin{bmatrix} \frac{a}{r} \\ \frac{b}{r} \\ \frac{c}{r} \\ \frac{d}{r} \end{bmatrix} = \begin{bmatrix} -p^2 - 3q^2 & 3p^2 + 9q^2 & 0 & 0 \\ p^2 + 3q^2 & 3p^2 + 9q^2 & 0 & 0 \\ 0 & 0 & (p^2 + 3q^2)p - 1 & (p^2 + 3q^2)3q - 3 \\ 0 & 0 & (p^2 + 3q^2)p - 1 & (p^2 + 3q^2)3q - 3 \end{bmatrix} * \begin{bmatrix} p \\ q \\ p \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

2.2. Programming Method (QDP Application)

We implemented an authentic way of finding quadruplets and we tried to take it further and also find some matrices that would enable us to build a quadruplet tree and traverse through all the quadruplets.

First of all we need to check and store the groups of numbers satisfying the relation we are looking for ($a^3 + b^3 + c^3 = d^3$) , because in order to find a relation, or a matrix (which is what we are aiming) we need at least 2 groups of numbers satisfying this property.

Thus we generated first 120 primitive quadruplets, found in Appendix B, that can help us study different properties they may have. First impression looking at them is quite interesting, because as the triplets have the triplet [3, 4, 5] as a root of the tree, our first quadruplet is

[3, 4, 5, 6].

After this step, we use the codes found in the Appendix C and go through different steps to check different properties we need them to have. First code creates 4 2D arrays by checking every possible element from -5 to 5 and saves the possible rows that can take us from the 1st quadruplet to the 2nd one. The boundary is somehow a must because we have an algorithm with a complexity $O(n^3)$ so more numbers will require a lot of time to process.

We use the 2nd algorithm to check every possible combination of the 2D arrays that we found at the 1st algorithm and we will save only 3 types of matrices: those with determinant = 0, 1 or -1, because this is what properties the matrices of the triplets' tree have.

Using the 2nd algorithm, we have saved all the matrices that had determinants as stated above and now we will use them to check if they can take us one step further, meaning if their 2nd step result is also a primitive quadruplet. So we multiply also the resulting quadruplet with the same matrix that we found it from and check the result if it is also a quadruplet.

CHAPTER 3

RESULTS AND DISCUSSION

3.1. Programming Method (QDP Application)

Among different results, some of the solutions were somehow more promising then the others giving us possibility to jump form one quadruplet to another but that would last only for 3-steps and then the resulting numbers would be close to a quadruplet but not a quadruplet.

One of the matrices that we found is very similar in properties to that of the Barning used in triplets. It has a determinant of 1 and a trace of 5

$$A = \begin{bmatrix} -1 & -1 & -2 & 3 \\ -3 & -2 & -5 & 8 \\ -3 & -2 & -1 & 5 \\ -4 & -2 & -5 & 9 \end{bmatrix}$$

Other matrices that we have found are described in the Appendix D where some of the test cases that we have done are written. Also its written the way they propagate and what quadruplets are found.

3.2. Tree of Quadruplets

Our main aim from the beginning of this study was to build a tree of quadruplets similar to that of the triplets.

In the tree shown below, we have used the quadruplet [3, 4, 5, 6] as root to generate other quadruplets based on similarity to the triplets' tree. Then we have generated 6 other quadruplets using different matrixes, as happens also with three matices that generate the next childrens.

Whats worth mentioning is that between two quadruplets in this tree there exists more than one matrix that enables us to go from one to the other. For example, just between [3, 4, 5, 6] and [1, 6, 8, 9] there are 5 matrices that yield [1, 6, 8, 9] when multiplied to the quadruplet

[3, 4, 5, 6]. All five of them yield the 3rd row of the tree.

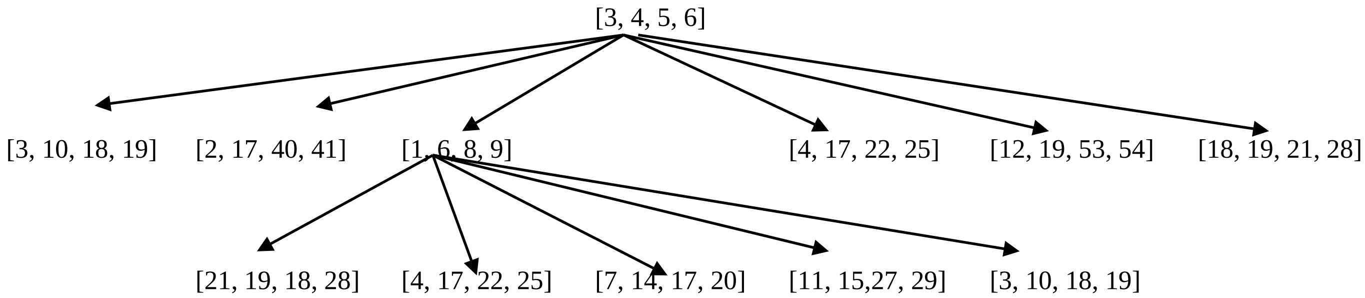

Figure 2. Found Tree Nodes of QDP

3.3. New Matrix Similar to Barning

Another interesting result is that when perfoming calculations using the barning method, wondering why some of the triplets are calculated as [21, 20, 29] and not as [20, 21, 29] we found 2 other 3x3 matrices that would find all the triplets that matrices used by Barning did but with a difference in the way the matrix traverses the tree.

$$D = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & 2 \\ -2 & 2 & 3 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

CONCLUSION

In this paper we tried to add something new to a very discussed and important topic among mathematicians and people focused on number theory. With our results we can conclude that there is more work to be done with the topic as the tools for solving problems have evolved. We really do hope that this paper serves as a start or a first step to pushing new researchers toward the beauty of number theory.

As a conclusion I really do think that in a near future there will be new discoveries to the tree of quadruplets with a bit of more computing power and maybe a more time and memory optimized algorithm.

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APPENDIX A

SVD decomposition of the Matrices used in the Pythagorean triplets

SVD	A	B	C
U	-0.5000 0.7071 -0.5000 -0.5000 -0.7071 -0.5000 -0.7071 0.0000 0.7071	-0.5000 0.7071 -0.5000 -0.5000 -0.7071 -0.5000 -0.7071 0.0000 0.7071	-0.5000 0.7071 -0.5000 -0.5000 -0.7071 -0.5000 -0.7071 0.0000 0.7071
S	5.8284 0 0 0 1.0000 0 0 0 0.1716	5.8284 0 0 0 1.0000 0 0 0 0.1716	5.8284 0 0 0 1.0000 0 0 0 0.1716
V	-0.5000 -0.7071 -0.5000 0.5000 -0.7071 0.5000 -0.7071 0.0000 0.7071	-0.5000 -0.7071 -0.5000 -0.5000 0.7071 -0.5000 -0.7071 0.0000 0.7071	-0.5000 -0.7071 -0.5000 -0.5000 0.7071 -0.5000 -0.7071 0.0000 0.7071

Table 3. SVD comparison of the three matrices found by Barning

APPENDIX B

Generation of the First 120 Primitive Quadruplets

Generating 120 result data took: 277266 mseconds

Sorted results:

0.	3	4	5	6	=>	27 + 64 + 125 = 216
1.	1	6	8	9	=>	1 + 216 + 512 = 729
2.	3	10	18	19	=>	27 + 1000 + 5832 = 6859
3.	7	14	17	20	=>	343 + 2744 + 4913 = 8000
4.	4	17	22	25	=>	64 + 4913 + 10648 = 15625
5.	3	18	24	27	=>	27 + 5832 + 13824 = 19683
6.	18	19	21	28	=>	5832 + 6859 + 9261 = 21952
7.	11	15	27	29	=>	1331 + 3375 + 19683 = 24389
8.	2	17	40	41	=>	8 + 4913 + 64000 = 68921
9.	16	23	41	44	=>	4096 + 12167 + 68921 = 85184
10.	5	30	40	45	=>	125 + 27000 + 64000 = 91125
11.	3	36	37	46	=>	27 + 46656 + 50653 = 97336
12.	27	30	37	46	=>	19683 + 27000 + 50653 = 97336
13.	29	34	44	53	=>	24389 + 39304 + 85184 = 148877
14.	12	19	53	54	=>	1728 + 6859 + 148877 = 157464
15.	7	42	56	63	=>	343 + 74088 + 175616 = 250047
16.	36	38	61	69	=>	46656 + 54872 + 226981 = 328509
17.	7	54	57	70	=>	343 + 157464 + 185193 = 343000
18.	14	23	70	71	=>	2744 + 12167 + 343000 = 357911
19.	38	43	66	75	=>	54872 + 79507 + 287496 = 421875
20.	31	33	72	76	=>	29791 + 35937 + 373248 = 438976
21.	19	60	69	82	=>	6859 + 216000 + 328509 = 551368
22.	28	53	75	84	=>	21952 + 148877 + 421875 = 592704

23.	50	61	64	85	=>	125000 + 226981 + 262144 = 614125
24.	38	48	79	87	=>	54872 + 110592 + 493039 = 658503
25.	20	54	79	87	=>	8000 + 157464 + 493039 = 658503
26.	21	43	84	88	=>	9261 + 79507 + 592704 = 681472
27.	25	31	86	88	=>	15625 + 29791 + 636056 = 681472
28.	17	40	86	89	=>	4913 + 64000 + 636056 = 704969
29.	58	59	69	90	=>	195112 + 205379 + 328509 = 729000
30.	19	53	90	96	=>	6859 + 148877 + 729000 = 884736
31.	45	69	79	97	=>	91125 + 328509 + 493039 = 912673
32.	11	66	88	99	=>	1331 + 287496 + 681472 = 970299
33.	12	31	102	103	=>	1728 + 29791 + 1061208 = 1092727
34.	13	51	104	108	=>	2197 + 132651 + 1124864 = 1259712
35.	15	82	89	108	=>	3375 + 551368 + 704969 = 1259712
36.	29	75	96	110	=>	24389 + 421875 + 884736 = 1331000
37.	16	47	108	111	=>	4096 + 103823 + 1259712 = 1367631
38.	50	74	97	113	=>	125000 + 405224 + 912673 = 1442897
39.	3	34	114	115	=>	27 + 39304 + 1481544 = 1520875
40.	23	86	97	116	=>	12167 + 636056 + 912673 = 1560896
41.	13	78	104	117	=>	2197 + 474552 + 1124864 = 1601613
42.	19	92	101	122	=>	6859 + 778688 + 1030301 = 1815848
43.	23	94	105	126	=>	12167 + 830584 + 1157625 = 2000376
44.	13	65	121	127	=>	2197 + 274625 + 1771561 = 2048383
45.	5	76	123	132	=>	125 + 438976 + 1860867 = 2299968
46.	86	95	97	134	=>	636056 + 857375 + 912673 = 2406104
47.	44	73	128	137	=>	85184 + 389017 + 2097152 = 2571353
48.	31	64	137	142	=>	29791 + 262144 + 2571353 = 2863288
49.	1	71	138	144	=>	1 + 357911 + 2628072 = 2985984
50.	71	73	138	150	=>	357911 + 389017 + 2628072 = 3375000
51.	46	47	148	151	=>	97336 + 103823 + 3241792 = 3442951

52.	17	102	136	153	=>	4913 + 1061208 + 2515456 = 3581577
53.	3	121	131	159	=>	27 + 1771561 + 2248091 = 4019679
54.	59	93	148	162	=>	205379 + 804357 + 3241792 = 4251528
55.	96	107	141	170	=>	884736 + 1225043 + 2803221 = 4913000
56.	54	80	163	171	=>	157464 + 512000 + 4330747 = 5000211
57.	19	114	152	171	=>	6859 + 1481544 + 3511808 = 5000211
58.	107	108	136	171	=>	1225043 + 1259712 + 2515456 = 5000211
59.	1	135	138	172	=>	1 + 2460375 + 2628072 = 5088448
60.	47	97	162	174	=>	103823 + 912673 + 4251528 = 5268024
61.	25	92	167	176	=>	15625 + 778688 + 4657463 = 5451776
62.	48	137	142	177	=>	110592 + 2571353 + 2863288 = 5545233
63.	17	57	177	179	=>	4913 + 185193 + 5545233 = 5735339
64.	108	109	150	181	=>	1259712 + 1295029 + 3375000 = 5929741
65.	68	113	166	185	=>	314432 + 1442897 + 4574296 = 6331625
66.	18	121	167	186	=>	5832 + 1771561 + 4657463 = 6434856
67.	58	131	160	187	=>	195112 + 2248091 + 4096000 = 6539203
68.	115	122	149	188	=>	1520875 + 1815848 + 3307949 = 6644672
69.	56	133	163	190	=>	175616 + 2352637 + 4330747 = 6859000
70.	34	123	173	192	=>	39304 + 1860867 + 5177717 = 7077888
71.	53	58	194	197	=>	148877 + 195112 + 7301384 = 7645373
72.	73	135	170	198	=>	389017 + 2460375 + 4913000 = 7762392
73.	27	46	197	198	=>	19683 + 97336 + 7645373 = 7762392
74.	6	127	180	199	=>	216 + 2048383 + 5832000 = 7880599
75.	45	53	199	201	=>	91125 + 148877 + 7880599 = 8120601
76.	81	147	167	203	=>	531441 + 3176523 + 4657463 = 8365427
77.	5	163	164	206	=>	125 + 4330747 + 4410944 = 8741816
78.	23	138	184	207	=>	12167 + 2628072 + 6229504 = 8869743
79.	113	146	166	209	=>	1442897 + 3112136 + 4574296 = 9129329
80.	42	83	205	210	=>	74088 + 571787 + 8615125 = 9261000

81.	56	61	210	213	=>	175616 + 226981 + 9261000 = 9663597
82.	58	157	179	214	=>	195112 + 3869893 + 5735339 = 9800344
83.	140	151	161	218	=>	2744000 + 3442951 + 4173281 = 10360232
84.	50	67	216	219	=>	125000 + 300763 + 10077696 = 10503459
85.	67	167	177	219	=>	300763 + 4657463 + 5545233 = 10503459
86.	108	163	170	219	=>	1259712 + 4330747 + 4913000 = 10503459
87.	31	95	219	225	=>	29791 + 857375 + 10503459 = 11390625
88.	18	167	193	228	=>	5832 + 4657463 + 7189057 = 11852352
89.	102	157	192	229	=>	1061208 + 3869893 + 7077888 = 12008989
90.	56	102	223	231	=>	175616 + 1061208 + 11089567 = 12326391
91.	103	140	204	231	=>	1092727 + 2744000 + 8489664 = 12326391
92.	85	107	220	232	=>	614125 + 1225043 + 10648000 = 12487168
93.	54	163	204	235	=>	157464 + 4330747 + 8489664 = 12977875
94.	96	163	198	235	=>	884736 + 4330747 + 7762392 = 12977875
95.	147	157	186	238	=>	3176523 + 3869893 + 6434856 = 13481272
96.	23	178	200	239	=>	12167 + 5639752 + 8000000 = 13651919
97.	113	124	220	241	=>	1442897 + 1906624 + 10648000 = 13997521
98.	73	174	207	244	=>	389017 + 5268024 + 8869743 = 14526784
99.	113	166	207	246	=>	1442897 + 4574296 + 8869743 = 14886936
100.	6	179	216	251	=>	216 + 5735339 + 10077696 = 15813251
101.	163	164	197	254	=>	4330747 + 4410944 + 7645373 = 16387064
102.	101	178	219	258	=>	1030301 + 5639752 + 10503459 = 17173512
103.	65	127	248	260	=>	274625 + 2048383 + 15252992 = 17576000
104.	29	174	232	261	=>	24389 + 5268024 + 12487168 = 17779581
105.	19	93	258	262	=>	6859 + 804357 + 17173512 = 17984728
106.	49	80	263	266	=>	117649 + 512000 + 18191447 = 18821096
107.	152	158	229	269	=>	3511808 + 3944312 + 12008989 = 19465109
108.	23	102	265	270	=>	12167 + 1061208 + 18609625 = 19683000
109.	31	186	248	279	=>	29791 + 6434856 + 15252992 = 21717639

110.	138	200	223	279	=>	2628072 + 8000000 + 11089567 = 21717639
111.	71	150	262	279	=>	357911 + 3375000 + 17984728 = 21717639
112.	50	172	257	281	=>	125000 + 5088448 + 16974593 = 22188041
113.	81	202	239	282	=>	531441 + 8242408 + 13651919 = 22425768
114.	113	116	271	284	=>	1442897 + 1560896 + 19902511 = 22906304
115.	71	177	262	288	=>	357911 + 5545233 + 17984728 = 23887872
116.	179	188	229	290	=>	5735339 + 6644672 + 12008989 = 24389000
117.	106	227	229	292	=>	1191016 + 11697083 + 12008989 = 24897088
118.	8	229	236	293	=>	512 + 12008989 + 13144256 = 25153757
119.	47	75	295	297	=>	103823 + 421875 + 25672375 = 26198073

APPENDIX C

Quadruplet Application Codes

I. We run and algorithm to find the sets of the matrixes that satisfy the multiplication from the fist set to the second set

```
v0=[3; 4; 5;6];
v1=[1; 6; 8;9];
r5(1, :)=[0 0 0 0];
l=0;
for i=-5:5
    m1=v1(1)-v0(1)*i;
    for j=-5:5
      m2=m1-v0(2)*j;
        for k = -5:5
            m3 = m2-v0(3)*k;
        if mod(m3, v0(4))==0
            l=l+1;
            r5(l, :)=[i  j  k m3/v0(4)];
          end
        end
    end
end
```

thus we have generated rows that satisfy the equations.

II. We run another algorithm in order to combine these rows so that we for matrices that satisfy the equations and also satisfying a determinant of 0,1 and -1 thus generating 3 matrixes

```
m=1;
B_0=zeros(4, 4, m);
for i=1:19
```

```
        for j=1:23
            for k=1:21
                for l = 1:22
                A=[r5(i, :); r6(j, :); r7(k, :);r8(l,:)];
                if det(A)==0
                    B_0(:, :, m)=A;
                    m=m+1;
                end
                end
            end
        end
    end
```

III. The 3rd step we run another algorithm to check for all of the 3 matrixes generated above if there exist matrixes that satisfy the 2nd step multiplication of the vectors we have

```
m_1=0;
n=0;
C_1(1,:)= zeros(1,3);
l =1 ;
for i=1:240
    vn=(B__1(:, :, i)*B__1(:, :, i))*v0;
      if
      vn(1)*vn(1)*vn(1)+vn(2)*vn(2)*vn(2)+vn(3)*vn(3)*vn(
      3)==vn(4)*vn(4)*vn(4)
        n=i;
        m_1=m_1+1;
        l = l+1;
        C_1(l,:)=[vn(1),vn(2),vn(3)];
    end
end
```

APPENDIX D

Test Cases and Matrices Found

Test cases including 0 in matrices

Between 3,4,5,6 and 4,17,22,25 (from -3 to 3)

1.
Det =0 , Trace = 2

$$\begin{matrix} 0 & 2 & -2 & 1 \\ 2 & -1 & -3 & 5 \\ 0 & 2 & -2 & 4 \\ 1 & -2 & 0 & 5 \end{matrix}$$

3,4,5,6
4,17,22,25
15,50,90,95 (3,10,18,19)
15,185,300,390 (not quadruplet)

2.

$$\begin{matrix} 0 & 3 & 2 & -3 \\ -2 & 2 & -3 & 5 \\ -2 & -3 & 2 & 5 \\ -2 & 1 & 3 & 2 \end{matrix}$$

3,4,5,6
4,17,22,25
20,85,110,125 (4,17,22,25)
100,425,550,625 (4,17,22,25)

3.

$$\begin{matrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 1 & 2 \\ -1 & -2 & -3 & 8 \\ 0 & 1 & 3 & 1 \end{matrix}$$

3,4,5,6
4,17,22,25
12,72,96,108 (3,18,24,27)
48,312,420,468 (not a quadruplet)

4.

2	2	-2	0
-3	3	-2	4
-2	-1	-2	7
-3	-1	-2	8

3,4,5,6
4,17,22,25
-2,95,106,127
-26,587,586,715 (not a quadruplet)

5.

3	2	1	-3
0	1	-1	3
3	1	-3	4
2	1	-3	5

3,4,5,6
4,17,22,25
-7,70,63,84 (-1,10,9,12)
-70,259,196,287 (not a quadruplet)

Between 3,4,5,6 to 1,6,8,9 including 0

Det -1

1.

0	-2	3	-1
1	2	-1	0
2	3	-2	0
2	2	-1	0

3,4,5,6
1,6,8,9
3,4,5,6
-4,9,13,12(not a quadruplet)

2.

0	0	-1	1
3	-2	1	0
0	0	-2	3
1	-1	2	0

3.

1	-1	-2	2
2	2	2	-3
-3	0	1	2
-3	-2	-2	6

4.

1	-1	-2	2
2	2	2	-3
1	1	-1	1
2	3	3	-4

5.

1	1	0	-1
1	3	3	-4
3	3	1	-3
2	1	1	-1

6.

2	-2	3	-2
1	0	-3	3
-3	0	1	2
-3	1	-2	4

7.

2	0	-1	0
1	1	1	-1
-1	2	3	-2
-1	-3	0	4

8.

3	-3	2	-1
0	1	-2	2
2	-1	0	1
2	0	-3	3

9.

3	1	0	-2
0	0	0	1
1	-1	-3	4
2	1	1	-1

Det = 1

1.

0	0	-1	1
3	-2	1	0
1	-2	-1	3
2	-3	3	0

2.

(loops btw 3,4,5,6 and 1,6,8,9)

0 1 3 -3

1 0 -3 3

2 2 0 -1

2 2 -1 0

3.

leads to 1,6,8,9 -> -2,9,15,16 no more.

1 -2 0 1

0 0 0 1

-1 1 -1 2

-1 1 -2 3

4. 1 -2 0 1

2 -3 0 2

1 0 1 0

1 -2 -2 4

5.

1 -1 -2 2

2 2 2 -3

-2 -1 0 3

-2 -3 -3 7

6.

1 -1 -2 2

2 2 2 -3

-1 0 1 1

-1 -2 -2 5